MATH 20510. Analysis in \mathbb{R}^n III (accelerated)

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1 Measure Theory

Definition 1.1. A family of sets A is called a *ring* if, for every $A, B \in A$,

- (i) $A \cup B \in \mathcal{A}$
- (ii) $A \setminus B \in \mathcal{A}$

Definition 1.2. A ring \mathcal{A} is called a σ -ring if for any $\{A_n\}_1^\infty \subseteq \mathcal{A}$,

$$\bigcup_{1}^{\infty} A_n \in \mathcal{A}.$$

Note. This implies that $\bigcap_{1}^{\infty} A_n \in \mathcal{A}$.

Definition 1.3. ϕ is a set function on a ring \mathcal{A} if for every $A \in \mathcal{A}$,

$$\phi(A) \in [-\infty, \infty].$$

Definition 1.4. A set function ϕ is *additive* if for any $A, B \in \mathcal{A}$ such that $A \cap B = \emptyset$,

$$\phi(A \cup B) = \phi(A) + \phi(B).$$

Definition 1.5. A set function ϕ is *countably additive* if for any $\{A_n\} \subseteq \mathcal{A}$ such that $A_i \cap A_j = \emptyset, \forall i \neq j$,

$$\phi\left(\bigcup_{1}^{n} A_{n}\right) = \sum_{1}^{n} \phi(A_{n}).$$

In the last two we assume that there are no $A, B \in \mathcal{A}$ such that $\phi(A) = -\infty, \phi(B) = \infty$.

Remark 1.6. If ϕ is an additive set function,

- (i) $\phi(\emptyset) = 0$.
- (ii) If A_1, \ldots, A_n are pairwise disjoint then $\phi(\bigcup_{1}^n A_n) = \sum_{1}^n \phi(A_n)$.
- (iii) $\phi(A_1 \cup A_2) + \phi(A_1 \cap A_2) = \phi(A_1) + \phi(A_2).$
- (iv) If ϕ is nonnegative and $A_1 \subseteq A_2$ then $\phi(A_1) \leq \phi(A_2)$.
- (v) If $B \subseteq A$ and $|\phi(B)| < \infty$ then $\phi(A \setminus B) = \phi(A) \phi(B)$.

Theorem 1.7. Let ϕ be a countably additive set function on a ring \mathcal{A} . Suppose $\{A_n\} \subseteq \mathcal{A}$ such that $A_1 \subseteq A_2 \subseteq \ldots$ and $A = \bigcup_1^{\infty} A_n \in \mathcal{A}$. Then $\phi(A_n) \to \phi(A)$ as $n \to \infty$.

Proof. Set $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$. Note

- (i) $\{B_n\}$ is pairwise disjoint.
- (ii) $A_n = B_1 \cup B_2 \cup \cdots \cup B_n$.
- (iii) $A = \bigcup_{1}^{\infty} B_n$.

Hence $\phi(A_n) = \sum_{1}^{\infty} \phi(B_j), \phi(A) = \sum_{1}^{\infty} \phi(B_j)$ and the conclusion follows. \Box

Definition 1.8. An *interval* $I = \{(a_i, b_i)\}_1^n$ of \mathbb{R}^n is the set of points $x = (x_1, \ldots, x_n)$ such that $a_i \leq x_i \leq b_i$ or $a_i < x_i \leq b_i$, etc. where $a_i \leq b_i$.

Note. \emptyset is an interval.

Definition 1.9. If A is the union of a finite number of intervals, we say A is *elementary*.

We denote the set of elementary sets by \mathcal{E} .

Definition 1.10. If I is an interval of \mathbb{R}^n , we define the volume of I by

$$\operatorname{vol}(I) = \prod_{i=1}^{n} (b_i - a_i)$$

If $A = I_1 \cup I_2 \cup \cdots \cup I_k$ is elementary, and the intervals are disjoint, then

$$\operatorname{vol}(A) = \sum_{1}^{k} \operatorname{vol}(I_j).$$

Remark 1.11. (i) \mathcal{E} is a ring, but not a σ -ring.

- (ii) If $A \in \mathcal{E}$, then A can be written as a finite union of disjoint intervals.
- (iii) If $A \in \mathcal{E}$, then vol(A) is well-defined.
- (iv) vol is an additive set function on \mathcal{E} , and vol ≥ 0 .

Definition 1.12. A nonnegative set function ϕ on \mathcal{E} is *regular* if $\forall A \in \mathcal{E}$, $\forall \varepsilon > 0, \exists$ open $G \in \mathcal{E}, G \supseteq A$ and closed $F \in \mathcal{E}, F \subseteq A$, such that

$$\phi(G) \le \phi(A) + \varepsilon, \qquad \phi(A) \le \phi(F) + \varepsilon.$$

Note. vol is regular.

Definition 1.13. A countable open cover of $E \subseteq \mathbb{R}^n$ is a collection of open elementary sets $\{A_n\}$ such that $E \subseteq \bigcup_1^\infty A_n$.

Definition 1.14. The Lebesgue outer measure of $E \subseteq \mathbb{R}^n$ is defined as

$$m^*(E) = \inf \sum_{1}^{\infty} \operatorname{vol}(A_n).$$

where \inf is taken over all countable open covers of E.

Remark 1.15. (i) $m^*(E)$ is well-defined.

- (ii) $m^*(E) \ge 0$.
- (iii) If $E_1 \subseteq E_2$ then $m^*(E_1) \le m^*(E_2)$.

Theorem 1.16. (i) If $A \in \mathcal{E}$, then $m^*(A) = \operatorname{vol}(A)$. (ii) If $E = \bigcup_{1}^{\infty} E_n$ then $m^*(E) \leq \sum_{1}^{\infty} m^*(E_n)$.

Proof.

Definition 1.17. Let $A, B \subseteq \mathbb{R}^n$.

(i) $A \triangle B = (A \setminus B) \cup (B \setminus A).$

- (ii) $d(A,B) = m^*(A \triangle B)$.
- (iii) We say $A_n \to A$ if $\lim_{n \to \infty} d(A_n, A) = 0$.

Definition 1.18. If there is a sequence of elementary sets $\{A_n\}$ such that $A_n \to A$ then we say A is *finitely m-measurable* and we write $A \in \mathfrak{M}_F(m)$.

Definition 1.19. If A is the countable union of finitely *m*-measurable sets, we say that A is *m*-measurable (Lebesgue measurable) and we write $A \in \mathfrak{M}(m)$.

Theorem 1.20. $\mathfrak{M}(m)$ is a σ -ring and m^* is countably additive on $\mathfrak{M}(m)$.

Definition 1.21. The *Lebesgue measure* is the set function defined on $\mathfrak{M}(m)$ by

$$m(A) = m^*(A), \quad \forall A \in \mathfrak{M}(m).$$

To summarize,

set function	domain	properties
vol	3	≥ 0 , additive, \mathcal{E} -regular
		$\geq 0, \ m^*(A) = \operatorname{vol}(A) \ \forall A \in \mathcal{E},$
m^*	$\subseteq \mathbb{R}^n$	countably subadditive
		$\geq 0, \ m(E) = m^*(E) \ \forall E \in \mathcal{M}(m),$
m	$\mathcal{M}(m)$	countable additivity(!)